Uncertainty and Investment Options

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Abstract

This paper develops a simple model in which uncertainty about future tax change leads to a temporary reduction in investment. When the uncertainty is resolved, investment recovers, generating a temporary boom.

Legislative bodies rarely act quickly, and during the period when a piece of new legislation is being formulated, there can be substantial uncertainty about its final form. If the legislation involves tax rates or other matters (trade policy, financial regulation) that affect the profitability of investments, this uncertainty increases the option value of delaying decisions.

This paper develops a simple model in which uncertainty about future tax policy leads to a temporary reduction in investment. The basic idea is that policy uncertainty creates uncertainty about the profitability of investment. If the uncertainty is likely to be resolved in the not-too-distant future, firms rationally delay committing resources to irreversible projects, reducing current investment. When the uncertainty is resolved, investment recovers, generating a temporary boom. The size of the boom
depends on the realization of the fiscal uncertainty, with lower realizations of the tax rate producing larger booms.

The model here is formulated in terms of tax policy and business investment, but the idea could as well be applied to business hiring decisions and household decisions about purchases of housing and other durables, and to uncertainty about financial regulation, trade policy, energy policy, and other matters that affect the profitability/desirability of various types of investment.

The mechanism studied here may be most important in prolonging and amplifying the consequences of other shocks. If a financial crisis produces a severe downturn, the private sector may wait for legislative decisions about whether to turn to fiscal stimulus, what form it will take, and how it will be financed. If the fiscal stimulus is ineffective, investors may wait again for decisions about a second round. If central bankers and political leaders stall on decisions about how to deal with a currency crisis or a potential default on sovereign debt, investors may choose to delay until the main outlines of a policy have been agreed upon.

In the model studied here investment has two inputs, projects and cash. Projects can be thought of as specific investment opportunities. For a retail chain or a service provider, projects might be cities or locations where new outlets could be built. For a manufacturing firm, a project might be the construction of a new plant. For a real estate developer, a project might be a parcel of land that could be built on. The key feature of a project is that it is an investment opportunity not available to others: it is exclusive to one particular investor. This feature is important in generating delay: the investor is willing to delay because he does not have to worry that someone else will exploit the opportunity if he waits.

Both projects and liquid assets can be stored, and delay is here defined as a situation where the stocks of both inputs are positive. When is delay an optimal strategy? In the model here, delay occurs if and only if there is uncertainty about future policy.
That is, uncertainty necessarily produces a period of delay, although that period may not occur immediately if the uncertainty is in the distant future. Conversely, anticipated policy changes can lead investors to store one input or the other, but unless there is uncertainty they will not store both.

Although there is a vast literature on investment under uncertainty, most of it focuses on uncertainty about idiosyncratic shocks to the demand for a firm’s product or to its cost of production. Recent work extends this literature to look at the aggregate effects of increased variance in these idiosyncratic shocks.¹

Most closely related to the model here are papers by Cukierman (1980) and Bernanke (1983). Cukierman looks at the decision problem of an individual firm with a single investment opportunity. The project is characterized by an unknown scale parameter, drawn from a known distribution. Each period the firm receives a signal about the parameter and updates its beliefs. The firm must decide when to invest—how long to wait and receive more information—and how much to invest. The paper shows that an increase in the variance of the distribution from which the parameter is drawn either increases or leaves unchanged the number of periods that investment is delayed.

Bernanke (1983) looks at a dynamic inference model, in which investment opportunities arrive every period and the underlying distribution from which these are drawn is, at random dates, replaced with a new one. When this happens, investors learn about it slowly, by observing the outcomes of previous investment decisions. There-

¹See Dixit and Pindyck (1994) and Stokey (2008) for more detailed discussions. More recently, Bloom (2009) and Arellano, Bai, and Kehoe (2011) develop aggregate models with idiosyncratic shocks to firm-level productivity. In these models more uncertainty means a higher variance for the distribution of shocks, and changes in the variance affects aggregate investment. In Bloom’s model the effects of more uncertainty come through fixed costs of investment, while in ABK they come through financing constraints. Lee (2012) looks at a setting in which investment opportunities must be created, and higher idiosyncratic volatility has a beneficial effect by inducing investors to create more opportunities and then set a high threshold for selection.
fore, after a switch occurs there is likely to be at least one period when investors are very uncertain which distribution is in place. The paper provides an example in which the switch from one distribution to another necessarily produces at least one period in which investors adopt a “wait and see” strategy and no investment takes place.

The empirical analysis in Baker, Bloom and Davis (2011) provides some confirmation of the idea that a temporary increase in policy uncertainty may depress investment. They construct an index of policy uncertainty that averages information from news media, the number of federal tax code provisions set to expire, and the extent of forecaster disagreement over future inflation and federal government purchases. Their VAR estimates show that an increase in policy uncertainty equal to the actual increase between 2006 and 2011 leads to a 13% decline in private investment. Fernández-Villaverde, et. al. (2011) look at a model with time-varying volatility and also find support for the adverse effect of an increase in fiscal volatility on economic activity.

The rest of the paper is organized as follows. Section 1 provides an overview of the model. In section 2 a benchmark model with no storage is studied, and two tax experiments are discussed. Section 3 sets out the full model in more detail, and studies the transitional dynamics after the announcement of a policy reform. The main result is Proposition 3, which shows that uncertainty about the new tax policy necessarily leads to delay. Section 4 looks at a numerical example. Section 5 extends the model to allow a Poisson arrival date, and Proposition 4 shows that the main result carries over, provided the arrival rate is not too small. Section 6 concludes.
1. OVERVIEW

The model uses an investment technology designed to produce an option value.\textsuperscript{2} Briefly, the key features are that each investment opportunity is specific to one firm, there is an intensity decision that is irreversible, and there are storage possibilities that permit delay. This rest of this section describes these features in more detail.

First, investment requires a project, as well as an input of cash. As noted above, projects should be thought of as specific investment opportunities. For retailers they might be new locations for outlets, for manufacturers they might be new plants, and so on. The key feature of a project is that it is available only to one particular investor. Hence that investor can wait to make a decision about how best to exploit it. This exclusivity assumption could be relaxed to some extent. For example, similar conclusions would hold if projects had a positive hazard rate of becoming available to other investors. The assumption cannot be dropped altogether, however. If a specific project were immediately available to multiple investors, there would be Bertrand-like competition to be the first to exploit it, precluding the possibility of delay.

Second, the intensity of investment in a project is an irreversible decision. Specifically, all investment in a particular project must take place at a single date: the capital cannot be increased or decreased later on. Thus, investment intensity has a putty-clay character: it is flexible ex ante but fixed ex post. This feature could also be relaxed. A model with costly reversibility would deliver similar conclusions, at the cost of added complexity.

Third, the firm can store projects. Here it is assumed that stored projects do not depreciate. A positive depreciation rate could be incorporated, but storage is key for creating an option value.

Fourth, the firm cannot borrow, but it can hold stocks of liquid assets. The inter-

\textsuperscript{2}It is similar to the “seeds” model in Jovanovic (2009).
est rate on these assets is assumed to be less than the rate used to discount dividend payments. Thus, in the absence of uncertainty holding liquid assets is unattractive, and the firm pays out dividends as quickly as possible. But in the presence of uncertainty, liquid assets can be attractive as a temporary investment while waiting for the uncertainty to be resolved. The no-borrowing assumption could be relaxed. If the interest rate on loans exceeded the interest rate used to discount dividends, the firm would borrow only when it needed additional funds for investment.

Finally, the total cost of investment is linear in the number of projects and strictly convex in the intensity. Strict convexity in the intensity is critical for making projects a valuable commodity. Without it, investment could be concentrated on a small set of projects, at no additional cost.

Formally, time is continuous, and new projects arrive at a constant rate $\mu$. At each date a firm chooses $n$, the number of projects (a flow), and $i$, the intensity of investment in each project. Total investment is the product $I = ni$ (a flow). The cost of implementing a project with intensity $i > 0$ is $g(i)$, where the function $g$ is strictly increasing and strictly convex. Hence cost minimization implies that the firm chooses the same intensity for all projects implemented at the same date, and investing at the total rate $I = ni$ has total cost $ng(i) = ng(I/n)$ (a flow) if it is allocated across $n$ projects. If projects and liquid assets have been accumulated, there may also be a discrete investment, where the number of projects $\hat{n}$, the increment to capital $\hat{ni}$, and the total cost $\hat{ng}(i)$ are masses. This type of investment is discussed in more detail later.

The role played by projects can be seen in a two-period example, $t = 1, 2$, with a project inflow of $\mu = 1$ in periods $t = 1, 2$, and $k_0$ given. The firm chooses the investment scale in each period, $0 \leq n_1 \leq 1$ and $0 \leq n_2 \leq 2 - n_1$, as well as the intensities, $i_1, i_2 \geq 0$. The capital stock and total investment costs are then

$$k_t = (1 - \delta) k_{t-1} + n_t i_t, \quad t = 1, 2,$$
This setup reduces to the usual model with convex investment costs if the firm chooses \( n_1 = n_2 = 1 \). But if the firm chooses to concentrate all investment in the second period, adjusting \( n_1 \) as well as \( i_1 \) reduces total costs. If total investment in the second period is \( I_2 > 0 \), total cost is reduced from \( g(I_2) \) to \( 2g(I_2/2) \). The model with projects maintains convexity in investment costs, but allows (limited) smoothing over time.

Installed capital \( k \) produces the revenue flow \( \pi(k) \) (net of variable costs) and depreciates at the constant rate \( \delta > 0 \). The revenue from installed capital is taxed at a flat rate \( \tau \), and uncertainty about \( \tau \) is the only risk the firm faces. We will study the effect of uncertainty about a one-time tax reform that is expected in the future. Two slightly different versions are considered. In the first version the date \( T \) of the tax reform is known, and only the new tax rate is uncertain. In the second version the date of the reform is also random, with a Poisson arrival time. In both cases the new tax rate \( \tau \) is drawn from a known distribution \( F(\cdot) \).

The following assumptions are maintained throughout:

—dividends are discounted at the constant rate \( \rho > 0 \);
—liquid assets held by the firm earn interest at the constant rate \( 0 \leq r < \rho \);
—the firm cannot borrow: all investment is from retained earnings, and
    the dividend must be nonnegative;
—capital cannot be sold: gross investment must be nonnegative;
—the firm receives a constant flow \( \mu > 0 \) of new projects;
—the revenue function \( \pi \) is strictly increasing, strictly concave, and twice differentiable, with \( \pi(0) = 0 \), \( \pi'(0) = \infty \), and \( \lim_{k \to \infty} \pi'(k) = 0 \);
—the cost function \( g \) is strictly increasing, strictly convex, and twice differentiable, with \( g(0) = 0 \), \( g'(0) \geq 0 \), and \( \lim_{i \to \infty} g'(i) = \infty \);
—the time horizon is infinite.
2. A BENCHMARK MODEL

We begin with the case where the date of the reform, call it \( T \), is known. In this section we study a benchmark model in which the firm does not have the option to store projects, although it can hold liquid assets. The full model is analyzed in the next two sections.

a. Optimal investment in the benchmark model

To develop the firm’s optimal strategy, we must begin with its problem at date \( T \), after the resolution of the uncertainty. From that time on, the tax rate is constant. Let \( \tau \geq 0 \) denote the tax rate and let \( k_T > 0 \) and \( a_T \geq 0 \) denote the firm’s stocks of capital and liquid assets at date \( T \). The firm can use part or all of the initial stock of assets to immediately pay a lump sum dividend, call it \( \hat{D} \). The firm’s problem is to choose the initial dividend \( \hat{D} \) and the subsequent dividend flow and investment intensity \( \{ D(t), i(t) \}_{t=T}^{\infty} \) to maximize the PDV of total dividends, discounted at the rate \( \rho > 0 \). Let \( w(k_T, a_T; \tau) \) denote the maximized value of the firm,

\[
w(k_T, a_T; \tau) \equiv \max \left[ \hat{D} + \int_T^{\infty} e^{-\rho(t-T)} D(t) \, dt \right] \tag{1}
\]

s.t. \[
\begin{align*}
\hat{a}_T &= a_T - \hat{D}, \\
\dot{k} &= \mu i - \delta k, \\
\hat{a} &= ra + (1-\tau) \pi(k) - D - \mu g(i), \\
0 &\leq \hat{D}, \hat{a}_T, D, i, a, \quad \text{all } t,
\end{align*}
\tag{2}
\]

where \( \hat{a}_T \) is the asset stock after the initial dividend.

As will be shown below, for any \( k_T > 0 \) and \( a_T \geq 0 \), the solution to (1)-(2) converges asymptotically to the unique steady state (SS), which has \( a^{ss} = 0 \) and

\[
(1-\tau) \pi'(k^{ss}) = (\rho + \delta) g'(i^{ss}), \quad \tag{3}
\]
\[ k^{ss} = \frac{\mu}{\delta} i^{ss}, \]
\[ D^{ss} = (1 - \tau) \pi(k^{ss}) - \mu g(i^{ss}). \]

It is straightforward to verify that \( k^{ss}, i^{ss}, \) and \( D^{ss} \) are strictly decreasing in \( \tau \).

Before proceeding, it is useful to bound the state space and the range for tax rates. Let \( \bar{k} = k^{ss}(0) \) be the SS capital stock when the tax is \( \tau = 0 \). Since we are interested in nonnegative tax rates, \( \bar{k} \) is a natural upper bound on the set of capital stocks to consider. Then define \( \tau > 0 \) as the tax rate for which investment to maintain the capital stock at \( \bar{k} \) just exhausts after-tax profits,

\[ (1 - \tau) \pi(\bar{k}) - \mu g(\delta \bar{k}/\mu) = 0. \]

For any lower tax rate and smaller capital stock, \( \tau \in [0, \bar{\tau}] \) and \( k \in (0, \bar{k}] \), after-tax revenue is sufficient to finance investment to offset depreciation.

Proposition 1 describes the solution to (1)-(2), which has a partial ‘bang-bang’ form with two critical values for capital, \( 0 < k_{crit}^1 < k^{ss} < k_{crit}^2 \leq \infty \). If the initial asset stock is zero, liquid assets are never acquired. While the capital stock \( k(t) \) is below the first critical value, all earnings are invested and the dividend is zero. While \( k(t) \) is above the second critical value, all earnings are paid out as dividends and there is no investment. In the intermediate region between the two critical values, both investment and the dividend are positive.

If initial assets are positive, \( a_T > 0 \), there are two cases. If \( k_T \geq k_{crit}^1 \), the entire asset stock is paid as an initial dividend, \( \dot{D} = a_T \), and the previous solution is unchanged. If \( k_T < k_{crit}^1 \), the initial dividend is less than initial assets, \( 0 \leq \dot{D} < a_T \), and the remainder of the assets are used to finance investment.

**Proposition 1:** For \( \tau \in [0, \bar{\tau}] \) and \( k_0 \in (0, \bar{k}] \), the solution to (1)-(2) has the following properties:

(a) If \( k_T < k^{ss} \), then \( k(t) \) and \( i(t) \) are strictly increasing along the transition path,
and there is a critical value \(0 < k_1^{\text{crit}} < k^{ss}\) such that \(D(t) = 0\) while \(k(t) \leq k_1^{\text{crit}}\), and \(D(t)\) is positive and strictly increasing while \(k(t) > k_1^{\text{crit}}\).

If \(k_T > k^{ss}\), then \(k(t)\) and \(D(t)\) are strictly decreasing along the transition path. If \(g'(0) > 0\), there is a critical value \(k_2^{\text{crit}} > k^{ss}\) such that \(i(t) = 0\) while \(k(t) > k_2^{\text{crit}}\), and \(i(t)\) is positive and strictly increasing while \(k(t) < k_2^{\text{crit}}\). If \(g'(0) = 0\), \(k_2^{\text{crit}} = +\infty\).

(b) If \(a_T = 0\), the stock of assets remains at zero throughout the transition, \(a(t) \equiv 0\).

If \(a_T > 0\) and \(k_T \geq k_1^{\text{crit}}\), the initial dividend is \(\hat{D} = a_T\), and the rest of the transition is as above.

If \(a_T > 0\) and \(k_T < k_1^{\text{crit}}\), there is a critical value \(\alpha(k_T) > 0\) for liquid assets with the following property. For \(0 < a_T < \alpha(k_T)\), there is no initial dividend: \(\hat{D} = 0\) and \(\hat{a}_T = a_T\). For \(a_T \geq \alpha(k_T)\), the initial dividend is \(\hat{D} = a_T - \alpha(k_T)\), and \(\hat{a}_T = \alpha(k_T)\). In either case the stock of assets is exhausted while \(k(t) < k_1^{\text{crit}}\) and remains at zero thereafter.

**Proof:** Let \(q_k, q_a\) denote the costate variables for capital and liquid assets. The solution \(\hat{D}\) and \(\{D, i, k, a, q_k, q_a, \ t \geq T\}\) satisfies

\[
1 \leq q_a(T), \quad \text{w/ eq. if } \hat{D} > 0,
\]

\[
1 \leq q_a, \quad \text{w/ eq. if } D > 0, \tag{4}
\]

\[
q_k \leq q_a g'(i), \quad \text{w/ eq. if } i > 0,
\]

\[
\dot{q}_k = (\rho + \delta) q_k - q_a (1 - \tau) \pi'(k), \tag{5}
\]

\[
\dot{q}_a \leq (\rho - r) q_a, \quad \text{w/ eq. if } a > 0, \quad \text{all } t \geq T,
\]

and the transversality conditions \(\lim_{t \to \infty} e^{-\rho t} q_x(t)x(t) = 0, \ x = k, a\). Clearly (3) is the steady state, where the assumptions on \(\pi\) and \(g\) ensure it is unique.

Suppose \(a_T = 0\), and conjecture the solution has \(a(t) \equiv 0\). The conjecture \(a(t) \equiv 0\) implies

\[
D = (1 - \tau) \pi(k) - \mu g(i), \quad \text{all } t \geq T. \tag{6}
\]
For any \((k, q_k)\), use (4) and (6) to define \(\chi(k)\) by

\[
\mu g \left[ g^{-1}(\chi(k)) \right] \equiv (1 - \tau) \pi(k).
\]

For any \(k\), the intensity \(i\) satisfying \(g'(i) = \chi(k)\) is just sufficient to absorb all of after-tax earnings. The function \(\chi\) is strictly increasing, with \(\chi(0) = g'(0)\), and the threshold \(\chi(k)\) divides the \((k, q_k)\)-space into two regions.

Above the threshold the dividend is zero and cash is at a premium: \(D = 0\) and \(q_o > 1\). In this region the firm is cash constrained. The investment intensity is determined by (6), so it is strictly increasing in \(k\) and independent of \(q_k\), and the second line in (4) determines \(q_o\).

Below the threshold \(D > 0\) and \(q_o = 1\). In this region the firm is not cash constrained. The investment intensity is determined by the second line in (4), so it is strictly increasing in \(q_k\) and independent of \(k\), and the dividend is determined by (6).

Hence the intensity isoquants in \((k, q_k)\) space are L-shaped, with kinks on the \((k, \chi(k))\) threshold. If \(g'(0) > 0\), there is a second threshold, the horizontal line where \(q_k = g'(0)\). Below this threshold \(i = 0\), and above it \(i > 0\).

The locus where \(\dot{k} = 0\) satisfies \(\mu \dot{i}(k, q_k) = \delta k\), so it is upward sloping, hitting the vertical axis at \(q_k = g'(0)\). The restrictions on \(\tau\) and \(k\) ensure that the \(\dot{k} = 0\) locus lies in the region where \(D > 0\), below \(\chi(k)\). The locus where \(\dot{q}_k = 0\) satisfies

\[
(\rho + \delta) g'[i(k, q_k)] = (1 - \tau) \pi'(k),
\]

so it is downward sloping in the region where \(D, i > 0\), and vertical in the regions where \(D = 0\) and \(i = 0\).

The stable manifold, call it \(SM_0\), slopes downward, and for any \(k_T \in (0, \overline{k})\) there is unique value \(q_{kT} > 0\) for which the system converges to the steady state. The critical values \(k_1^{\text{crit}} < k^{ss} < k_2^{\text{crit}}\) are defined by the points where \(SM_0\) cuts the \(\chi(k)\) and \(g'(0)\) thresholds. Along \(SM_0\), intensity \(i\) increases with \(k\) (as \(q_k\) falls) for \(k < k_1^{\text{crit}}\), decreases
with \( k \) for \( k \in (k_1^{\text{crit}}, k_2^{\text{crit}}) \), and is constant at \( i = 0 \) for \( k \geq k_2^{\text{crit}} \). The dividend is \( D = 0 \) for \( k \leq k_1^{\text{crit}} \) and increases with \( k \) for \( k > k_1^{\text{crit}} \).

To verify that the conjecture \( a(t) \equiv 0 \) is correct, it suffices to show that \( \dot{q}_a/q_a \leq \rho-r \). If \( D > 0 \), then \( q_a = 1 \) and \( \dot{q}_a = 0 \). If \( D = 0 \), then \( k < k^{ss} \), so \( k \) and \( i \) are increasing, and \( q_k \) is decreasing. Since \( q_a = q_k/g'(i) \), it follows that \( \dot{q}_a < 0 \).

If \( a_T > 0 \) and \( k_T \geq k_1^{\text{crit}} \), then \( \dot{D} = a_T, \dot{a}_T = 0 \), and the rest of the solution is as above.

For \( a_T > 0 \) and \( k_T < k_1^{\text{crit}} \), solutions can be constructed as follows. Choose any point \((k, q_k)\) on \( SM_0 \) with \( k < k_1^{\text{crit}} \), use (6) with \( D = 0 \) to determine \( i \), and calculate \( q_a = q_k/g'(i) > 1 \). Construct trajectories for \((k, a, q_k, q_a)\) by running the ODEs in (2) and (5) backward in time, using \( \dot{q}_a/q_a = \rho - r > 0 \) and \( g'(i) = q_k/q_a \). The restriction \( q_a \geq 1 \) limits the length of the extension. The terminal pairs \((k_T, \alpha(k_T))\) for the longest extensions define the function \( \alpha \).

Varying the length of the backward extension traces out a one-dimensional family of initial conditions \((k_T, a_T)\), and varying the initial point on \( SM_0 \) gives a two-dimensional family. Lower initial values for \( k \) on \( SM_0 \) have higher initial values for \( q_a \), allowing longer extensions. Hence \( \alpha \) is a continuous, decreasing function, with \( \alpha(k) \to 0 \) as \( k \uparrow k_1^{\text{crit}} \).

For \( 0 < a_T \leq \alpha(k_T) \), the initial dividend is \( \dot{D} = 0 \), and the transition begins with \( \dot{a}_T = a_T \) and \( q_{aT} > 1 \). The solution follows a constructed trajectory until assets are exhausted. This occurs while \( k < k_1^{\text{crit}} \), and thereafter the solution follows \( SM_0 \). For \( a_T > \alpha(k_T) \), the initial dividend is \( \dot{D} = a_T - \alpha(k_T) \), and the transition begins with \( \dot{a}_T = \alpha(k_T) \) and \( q_{aT} = 1 \). ■

Figure 1 shows the phase diagram in \((k, q_k)\)–space, where \( a(t) \equiv 0 \) and \( q_a \) is not displayed. Above the threshold \( \chi(k) \), the broken curve, the firm is cash constrained, with \( D = 0 \) and \( q_a > 1 \). The \( \dot{k} = 0 \) and \( \dot{q}_k = 0 \) loci are the dotted curves, and the stable manifold \( SM_0 \) is the solid curve. The values \( \overline{k} \equiv k^{ss}(0) \) and \( \underline{k} \equiv k^{ss}(\overline{r}) \) are
indicated on the horizontal axis. The critical value $k_{1}^{\text{crit}}$ is defined by the intersection of $SM_0$ and $\chi(k)$. In this example $k_{2}^{\text{crit}} \gg k$, so the region where $i = 0$ does not appear in the figure.

b. Tax experiments in the benchmark model

Next consider two tax experiments in this benchmark model. In each case the initial tax rate is $\tau \in [0, \bar{\tau}]$, and at $t = 0$ the firm is at the associated steady state, $(k_0, a_0) = (k^{ss}(\tau), 0)$. The first experiment consists of a permanent change in the tax rate, to $\hat{\tau}$, that is announced at $t = 0$ and takes effect at $T > 0$. The second experiment is like the first except that the new tax rate is uncertain, and at $t = 0$ only its distribution $F(\hat{\tau})$ is known.

The first experiment is a special case of the second, where $F$ puts unit mass on a single point, so in either case the firm’s problem is to choose $\{D, i\}_{t=0}^{T}$ to solve

$$\max \int_{0}^{T} e^{-\rho t} D(t) dt + e^{-\rho T} E_{\hat{\tau}} [w(k(T), a(T); \hat{\tau})]$$

subject to the constraints in (2), where the continuation value $w(\cdot)$ is the function defined in (1) and the expectation uses the distribution $F$. Since $a_0 = 0$, there can be no initial dividend.

The solution satisfies (4) and (5), as before, but the terminal conditions are now

$$\lim_{t \uparrow T} q_k(t) = E_{\hat{\tau}} [w(k(T), a(T); \hat{\tau})] = E_{\hat{\tau}} [q_kT(\hat{\tau})],$$

$$\lim_{t \uparrow T} q_a(t) \geq E_{\hat{\tau}} [w_a(k(T), a(T); \hat{\tau})] = E_{\hat{\tau}} [q_aT(\hat{\tau})], \quad \text{w/ eq. if } a(T) > 0,$$

where $q_{xT}(\hat{\tau})$, $x = k, a$, are the initial costate values for the problem in (1), given initial values $(k_T, a_T) = (k(T), a(T))$ for the states and tax rate $\hat{\tau}$. Thus, the costate for capital approaching date $T$, before the uncertainty is resolved, must equal its expected value ex post. The costate for liquid assets must equal or exceed its value ex post, with equality required only if the stock of assets is strictly positive.
In either experiment the adjustment begins in anticipation of the tax change. Consider first deterministic changes. The adjustment to a tax increase (of any size) or to a modest tax decrease does not involve accumulating liquid assets. To see this, conjecture that \( a(t) \equiv 0 \) and consider the phase diagram in \((k, q_k)\)-space. Since (8) implies \( q_k \) is continuous at date \( T \), the pair \((k(T), q_k(T))\) must lie on \( SM_0 \) for the new tax rate \( \hat{\tau} \). Working backward, this requirement determines the direction and size of the jump in \( q_k \) at \( t = 0 \). To verify the conjecture, it suffices to show that \( q_a \) satisfies the second lines in (5) and (8).

Figure 2a shows two transitions, for a tax increase and a decrease. The initial rate in each case is \( \tau^M \), and the new rates are \( \tau^H, \tau^L \), with \( \tau^H > \tau^M > \tau^L \).

For a tax increase, \( q_k \) jumps down at \( t = 0 \), and both \( k \) and \( q_k \) decline over \((0, T)\). At date \( T \) the transition reaches the stable manifold for the new tax rate, call it \( SM^H \), and the rest of the transition follows that trajectory, with \( k \) continuing to decline while \( q_k \) rises. Since the transition over \((0, T)\) lies below the locus where \( \dot{k} = 0 \) for the initial tax rate \( \tau^M \), it is also below the threshold \( \chi(\cdot; \tau^M) \). In this region \( D > 0 \), \( q_a = 1 \), and \( \dot{q}_a/q_a = 0 \), so (5) and (8) hold, verifying the conjecture that \( a(t) \equiv 0 \).

For a tax decrease, the dynamics are reversed and the transition over \((0, T)\) lies above \( \dot{k} = 0 \). Nevertheless, if the decrease is not too large, as in Figure 2a, the path lies below the threshold \( \chi(\cdot; \tau^M) \), and the previous argument applies.

If the tax decrease is sufficiently large, however, the transition over \((0, T)\) can lie partly or entirely in the region above \( \chi(\cdot; \tau) \), where \( q_a > 1 \) and the constraint \( \dot{q}_a/q_a \leq \rho - r \) may bind. In general, the solution then has two phases, with no dividend paid in either phase. On \((0, T_0)\) no assets are held. Intensity \( i(t) \) is determined by the cash constraint, the costate for assets is determined by \( q_a = q_k/g'(i) \), and the latter must satisfy \( \dot{q}_a/q_a \leq \rho - r \). On \((T_0, T)\) the firm first acquires and then spends liquid assets, using them to boost the investment intensity as date \( T \) draws near. In this phase the costate is determined by \( \dot{q}_a/q_a = \rho - r \), the intensity satisfies \( g'(i) = q_k/q_a \),
and assets evolve according to the law of motion in (2). Assets rise and fall over this phase, with \( a(T_0) = a(T) = 0 \). Clearly the requirement \( q_a(T) \geq 1 \) does not bind.

Note that a one-phase solution, with either phase, is possible. If the tax cut is only moderately large, the firm uses all available funds for current investment over the whole interval \((0, T)\). The requirement for a solution of this type is that \( q_a \) cannot rise too quickly along the transition path. At the other extreme, if the tax cut is very large and \((\rho - r)\) is small, the firm may start acquiring assets immediately after the announcement, so \( T_0 = 0 \).

For a stochastic tax change the argument is similar except that the transition involves two jumps in the costates and controls, one at \( t = 0 \) when the change is announced, and another at \( T \) when the uncertainty about the rate is resolved. At date \( T \), given \( k_T \) and the realized value \( \hat{\tau} \) for the tax rate, the costate jumps so the pair \((k_T, q_{kT}(\hat{\tau}))\) lies on \( SM(\hat{\tau}) \). The values \( q_{kT}(\hat{\tau}) \) must jointly satisfy (16), and the direction and size of the jump in \( q_k \) at \( t = 0 \) is determined by this requirement.

As before, if the tax change involves increases (of any size) or decreases that are not too large, no liquid assets are accumulated. If \( F \) puts sufficiently high probability on a set of large tax decreases, however, liquid assets may be acquired and then spent over \((0, T)\), as in the deterministic case.

Figure 2b shows an example where the initial tax rate is \( \tau^M \) and the support of \( F \) consists of the two points \( \tau^L \) and \( \tau^H \). The example, \( F \) puts a high probability on \( \tau^H \), so \( q_k \) jumps down at \( t = 0 \). Both \( k \) and \( q_k \) decline over \((0, T)\), so the transition lies below \( \chi(\cdot; \tau^M) \), in the region where \( q_a = 1 \). At date \( T \) the costate jumps again, so the pair \((k_T, q_{kT})\) lies on \( SM(\hat{\tau}) \), and the rest of the adjustment follows that trajectory.

Figure 3 show the short-run dynamics for a tax cut anticipated one year in advance. The dashed lines show the adjustment when the new rate is known, and the solid lines the adjustment if it uncertain. Uncertainty slows down the adjustment, but the qualitative behavior of the transition is not much changed.
3. PROJECTS AS OPTIONS

The full model adds one new feature: projects can be stored. Thus, a project is an option to invest that can be exploited immediately or held for future use. The state variable for the firm is then \( s = (k, a, m) \), where \( m \) is the stock of projects, and its controls are \((D, i, n)\), where \( n \) is the flow of projects actually utilized.

a. Optimal investment in the option model

As before, we will first consider the firm’s problem after the tax reform. Let \( \hat{\tau} \) denote the post-reform tax rate, and consider the optimal investment path from date \( T \) for a firm with initial stocks \( k_T > 0 \) and \( a_T, m_T \geq 0 \). If \( a_T, m_T > 0 \), the firm can make a one-time discrete adjustment (DA), using some or all of its stocks of cash and projects to produce an increment to its capital stock. In particular, it can invest in a mass of projects \( \hat{n} \geq 0 \), with intensity \( \hat{i} \geq 0 \), producing a mass of new capital goods \( \hat{I} = \hat{n}\hat{i} \). The cost of this investment, \( \hat{n}g(\hat{i}) \), is financed out of liquid assets. As before, it can also use liquid assets to pay a discrete dividend \( \hat{D} \). After these one-time adjustments, if any, the firm faces a standard control problem.

Let \( v(s_T; \hat{\tau}) \) denote the maximized value of the firm. Given \( s_T \) and \( \hat{\tau} \), the firm chooses \( (\hat{D}, \hat{i}, \hat{n}) \) and \( \{(D, i, n)\}_{t=T}^{\infty} \) to solve

\[
v(s_T; \hat{\tau}) \equiv \max \left[ \hat{D} + \int_T^{\infty} e^{-\rho(t-T)} D(t) dt \right]
\]

s.t. 
\[
\begin{align*}
\dot{k}_T &= k_T + \hat{n}\hat{i}, \\
\dot{a}_T &= a_T - \hat{D} - \hat{n}g(\hat{i}), \\
\dot{m}_T &= m_T - \hat{n}, \\
0 &\leq \hat{D}, \hat{i}, \hat{n}, \hat{a}_T, \hat{m}_T, \\
\dot{k} &= n\hat{i} - \delta k,
\end{align*}
\]

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\[ \dot{\alpha} = r\alpha + (1 - \hat{\tau})\pi(k) - D - ng(i), \]
\[ \dot{\mu} = \mu - n, \]
\[ 0 \leq D, i, n, a, m, \quad \text{all } t > T, \]

where \( \hat{s}_T = (\hat{k}_T, \hat{a}_T, \hat{m}_T) \) denotes the firm’s state after the DA. Letting \( q_m \) denote the costate for \( m \), the discrete adjustment \( (\hat{D}, \hat{i}, \hat{n}) \) satisfies\(^3\)

\[ 1 \leq q_{aT}, \quad \text{w/ eq. if } \hat{D} > 0, \]
\[ q_{kT} \leq q_{aT}g'(\hat{i}), \quad \text{w/ eq. if } \hat{n}i > 0, \]
\[ q_{kT}\hat{i} \leq q_{mT} + q_{aT}g(\hat{i}), \quad \text{w/ eq. if } \hat{n}i > 0, \]

where \( q_{kT}, q_{aT}, \) and \( q_{mT} \) are the costate values at date \( T \), after \( \hat{\tau} \) is realized. Thereafter the solution satisfies

\[ 1 \leq q_{a}, \quad \text{w/ eq. if } D > 0, \]
\[ q_{k} \leq q_{a}g'(\hat{i}), \quad \text{w/ eq. if } n\hat{i} > 0, \]
\[ q_{k}\hat{i} \leq q_{m} + q_{a}g(\hat{i}), \quad \text{w/ eq. if } n\hat{i} > 0, \quad \text{all } t > T, \]

\[ \dot{q}_{k} = (\rho + \delta) q_{k} - q_{a} (1 - \hat{\tau})\pi'(k), \]
\[ \dot{q}_{a} \leq (\rho - r) q_{a}, \quad \text{w/ eq. if } a > 0, \]
\[ \dot{q}_{m} \leq \rho q_{m}, \quad \text{w/ eq. if } m > 0, \quad \text{all } t > T, \]

and the transversality conditions \( \lim_{t \to \infty} e^{-\rho t} q_x(t)x(t) = 0, x = k, a, m. \) Note that the SS values \( (D^{ss}, i^{ss}, k^{ss}, a^{ss}) \) for the model with optional investment are the same as in the benchmark model, with \( n^{ss} = \mu, \) and \( m^{ss} = 0. \)

The following proposition describes one key feature of the dynamics in the model where projects are options.

\(^3\)See Kamien and Schwartz (1991) or Seierstad and Sydsæter (1977) for a detailed discussion of these conditions.
**Proposition 2:** For any $(k_T, a_T, m_T)$ and any $\hat{\tau}$, the solution to the problem in (9)-(11) has the property that the discrete adjustment $\left(\hat{D}, \hat{i}, \hat{n}\right)$ exhausts at least one of the input stocks: $\hat{n} = m(T)$ or $\hat{n}g(\hat{i}) + \hat{D} = a(T)$ or both.

The proof is in Appendix A.

**b. Tax changes in the option model**

Next consider the two tax experiments. As before, suppose that at $t = 0$ the firm is at the steady state for the tax rate $\tau$, so $(k_0, a_0, m_0) = [k^{ss}(\tau), 0, 0]$. The new rate, which takes effect at $T > 0$, is drawn from the known distribution $F(\hat{\tau})$, and there are no changes thereafter.

Since there are no initial stocks of liquid assets or projects, there can be no DA at $t = 0$. Hence the firm chooses $\{ (D, i, n) \}_{t=0}^T$ to solve

$$\max \int_0^T e^{-\rho t} D(t) dt + e^{-\rho T} E_{\hat{\tau}} \left[ v(s(T); \hat{\tau}) \right] \quad \text{s.t. (11)}. \quad (15)$$

The conditions in (13)-(14) are again necessary for an optimum and, using the same reasoning as in (8), the terminal conditions for the costates are

$$\lim_{t \uparrow T} q_k(t) = E_{\hat{\tau}} \left[ q_kT(\hat{\tau}) \right], \quad (16)$$

$$\lim_{t \uparrow T} q_x(t) \geq E_{\hat{\tau}} \left[ q_xT(\hat{\tau}) \right], \quad \text{w/ eq. if } x_T > 0, \quad x = a, m.$$}

**Deterministic tax changes.**—

Suppose $F$ puts unit mass on a single point, so the new tax rate is known at $t = 0$. Then the solutions for the option and benchmark models can differ if the tax change is sufficiently large, but they coincide for small or moderate changes.

For a sufficiently large tax increase, the benchmark model has $i = 0$ on $(0, T)$ and unused projects are discarded. In the option model those projects are accumulated and the subsequent transition is altered.
In all other cases the benchmark solution has $n = \mu$ and $i > 0$ on $(0, T)$. Hence $q_k = q_a g'(i)$ and

\[
q_m = q_k i - q_a g(i),
\]
\[
\dot{q}_m = \dot{q}_k i - \dot{q}_a g(i), \quad t \in (0, T).
\]

Thus, the solution for the benchmark model is also a solution for the option model if

\[
\frac{\dot{q}_m}{q_m} \leq \rho \quad \text{and} \quad \lim_{t \uparrow T} q_m(t) \geq q_{mT}(\hat{\tau}). \tag{17}
\]

For any tax increase, the benchmark transition stays in the region where $q_a = 1$ and $\dot{q}_a = 0$, so $\dot{q}_m = \dot{q}_k i$. For a tax increase $q_k$ and $q_m$ are falling over $(0, T)$, so the first condition in (17) holds. In addition, $q_a$ and $i$ are continuous at $T$ in the benchmark transition, so $q_m$ is also continuous, and the second condition in (17) holds as well. Hence the benchmark transition is also a solution for the option model.

The same is true for tax decreases that are not too large. Two conditions limit the size of the tax cut. First, it must be modest enough so that the benchmark transition stays in the region where $q_a = 1$. In addition, since $q_k$ and $q_m$ are rising over $(0, T)$ for a tax decrease, the size of the cut must be small enough so that first condition in (17) holds.

If the benchmark solution on $(0, T)$ lies in the region where $q_a > 1$, the option and benchmark solutions necessarily differ. In the benchmark solution the firm is cash constrained on $(0, T)$. At date $T$ the investment intensity jumps, since the tax cut increases the profit flow and loosens the cash constraint. In the option model the firm holds some projects in inventory before $T$ and uses them after $T$, smoothing the intensity. [For a very large tax cut and $(r + \delta) / (\rho + \delta)$ is close to unity, it seems possible that both inputs are held before $T$.] But even if the benchmark solution lies in the region where $q_a = 1$, so no assets are acquired, the first condition in (17) may nevertheless fail if the tax cut is large.
Stochastic tax changes.—

If there is uncertainty about the new tax rate, the optimal response always involves a period of delay: there is an interval of time before $T$ during which investment ceases.

**Proposition 3:** Suppose a tax change at $T > 0$, drawn from the distribution $F$, is announced at $t = 0$. Unless $F$ puts unit mass at a single point, there exists $\Delta > 0$ such that $n(t)i(t) = 0$, for $t \in (T - \Delta, T)$.

The proof, in the Appendix, is by contradiction. Suppose the contrary. Because $g$ is convex, smoothing the intensity of investment across projects reduces the total cost. Delaying some projects from before $T$ until just after $T$ permits this type of smoothing. Of course, if the uncertainty is small in magnitude, the period of delay is short. Thus, if $T$ is large, the period of delay may not begin at $t = 0$.

The next two sections look at the transition in more detail.

c. The DA and post-DA transition

The DA and the post-DA transition depend on the realization of $\hat{\tau}$. Figure 3 shows, qualitatively, how the discrete adjustment $[\hat{n}, \hat{i}, \hat{ng}(\hat{i}), \hat{D}]$ and the initial costate value $\hat{q}_{aT}$ vary with $\hat{\tau}$. As shown there, the support of $F$ can be divided into three regions, with different qualitative behavior for the transition. Note that the description of $\hat{\tau}$ as low, moderate or high means relative to other values in the support of $F$. The initial tax rate $\tau$ may be higher or lower than all these values.

For low realizations of $\hat{\tau}$, Region A in Figure 3, the firm is cash constrained. The initial investment exhausts the firm’s stock of liquid assets, $\hat{ng}(\hat{i}) = a_T$, but some projects remain, $\hat{n} < m_T$. No initial dividend is paid, $\hat{D} = 0$, and cash is at a premium, $\hat{q}_{aT} > 1$. Let $\Delta_T > 0$ denote the length of time required to exhaust the remaining stock of projects. During this period no dividend is paid, and all earnings are used for investment. At $T + \Delta_T$ the solution lies on $SM_0(\hat{\tau})$, and the remaining
transition is as in the benchmark model. In this region \( \hat{n} \) is strictly increasing in \( \hat{\tau} \), while \( \hat{i} \), \( \hat{q}_{aT} \) and \( \Delta_T \) are strictly decreasing. At the threshold value for \( \hat{\tau} \) separating this region from the next, \( \hat{n} = m_T \), \( \hat{q}_{aT} = 1 \), and \( \Delta_T = 0 \).

For moderate realizations of \( \hat{\tau} \), Region B in Figure 3, the initial investment uses the entire stock of projects, \( \hat{n} = m_T \), and the firm has more than enough assets to finance investment at the desired intensity. Excess cash remains and is paid as a dividend, \( \hat{D} = a_T - \hat{n}g(\hat{i}) > 0 \), which implies \( \hat{a}_T = 0 \) and \( \hat{q}_{aT} = 1 \). The DA puts the state on \( SM_0(\hat{\tau}) \), and the rest of the transition is as in the benchmark model. Since \( q_{aT} = 1 \), the post-DA capital stock must satisfy \( \hat{k}_T \geq k_{1_{\text{crit}}}^\text{crit}(\hat{\tau}) \).

For high realizations of \( \hat{\tau} \), Region C in Figure 3, neither the stock of projects nor the stock of liquid assets is exhausted by the initial investment, and the excess assets are paid as a dividend. That is, \( \hat{n} < m_T \), \( \hat{D} = a_T - \hat{n}g(\hat{i}) > 0 \), and \( \hat{q}_{aT} = 1 \). Indeed, for \( \hat{\tau} \) sufficiently large, \( \hat{n} = 0 \). The post-DA adjustment starts with a stock of projects \( \hat{m}_T > 0 \), which is used up over some period of time \( \Delta_T \), at which point \( (k, q_k) \) lies on \( SM_0(\hat{\tau}) \).

One important feature of the solution is clear from Figure 3: the firm’s optimal strategy before \( T \) necessarily produces a positive probability of being cash constrained when \( \hat{\tau} \) is realized. To see this, note that since liquid assets are acquired before \( T \), it follows that \( q_a \) is increasing on \( (0, T) \), so \( q_{aT} > 1 \). At date \( T \), the post-realization value satisfies \( \hat{q}_{aT} > 1 \) only if the new tax rate lies in Region A. Hence (16) implies that the solution lies in Region A—where the firm is cash constrained—with strictly positive probability.

In addition, notice that while the intensity \( \hat{i} \) of the discrete investment is decreasing in \( \hat{\tau} \) over the entire range, the scale \( \hat{n} \) of that investment is increasing in Region A, constant in Region B, and decreasing in Region C. Thus, a stock of projects remains after the DA in Regions A and C. The economic motivation for holding investment options after \( T \) is different in those two regions, however. In Region A the firm is
accumulating capital, but it is cash constrained. Thus, it holds some projects back in order to finance them later, out of retained earnings, at higher intensities. In Region C the firm is decumulating capital, and it is profitable to hoard some projects to reduce the cost of replacement investment later on.

d. The period of delay

Next consider the firm’s optimal strategy over \((0, T)\), when it is accumulating stocks of projects and liquid assets. Since \(a_T, m_T > 0\), all three conditions in (16) must hold with equality. The size of the stocks that the firm accumulates can be determined as follows.

Suppose \(T\) is not too large, and conjecture that \(n(t) = 0\) on \((0, T)\). Then

\[
k_T = k^{ss}(\tau)e^{-\delta T},
\]

\[
m_T = \mu T.
\]

Since \(r < \rho\), the firm does not pay a dividend while it holds liquid assets. Thus, the strategy for acquiring liquid assets is to choose a date \(S \in (0, T)\). Before date \(S\), all earnings are paid as dividends, and after \(S\) all earnings are retained. Given \(S\), the stock of liquid assets and the value of the costate \(q_a\) at \(T\) are

\[
a_T = \int_S^T (1 - \tau) \pi \left[ k^{ss}(\tau)e^{-\delta t} \right] e^{r(T-t)} dt > 0,
\]

\[
q_aT = e^{(\rho-r)(T-S)} > 1.
\]

Thus, increasing \(S\) reduces both \(a_T\) and \(q_aT\).

When the new tax rate is realized, a larger initial stock of liquid assets \(a_T\) reduces \(\hat{q}_aT(\hat{\tau})\) in region A, and shrinks the size of that region. Hence \(E_{\hat{\tau}}[\hat{q}_aT]\) is a decreasing function of \(a_T\), and there is at most one value \(S\) for which \(E_{\hat{\tau}}[\hat{q}_aT] = q_aT\). If no solution exists, the conjecture that \(n(t) = 0\) on \((0, T)\) is incorrect, and a smaller value for \(T\) is needed.
For the other two costates, use \( \theta \) to determine \( q_{kT} \) and \( q_{mT} \), and then use the laws of motion in (14) to construct solutions over \((0, T)\).

4. AN EXAMPLE

The example uses the revenue and cost functions

\[
\pi(k) = Ak^\alpha, \quad g(i) = i + \frac{1}{2}g_2i^2,
\]

and the parameter values

\[
A = 1, \quad \alpha = 0.70, \quad g_2 = 1.5, \quad \delta = 0.10, \\
\mu = 1, \quad \rho = 0.04, \quad r = 0.03.
\]

The initial tax rate is \( \tau = \tau^L = 0.20 \), and the tax cut is anticipated \( T = 2 \) years in advance. The post-reform rate is

\[
\hat{\tau} = \begin{cases} 
\tau^M = 0.22, & \text{with probability } 0.566, \\
\tau^H = 0.40, & \text{with probability } 0.434.
\end{cases}
\]

Thus, the tax reform could raise the tax rate by 2 or 20 percentage points.

The short run consequences of the policy are shown in Figure 5. First consider the interval \((0, T)\). At date \( t = 0 \) the firm is at the steady state capital stock for the initial tax rate, with levels for all the displayed variables indicated by black diamonds on the vertical axes in each panel. At \( t = 0 \) the firm stops investing, and \( ni = 0 \) over \((0, T)\), as shown in panel (a). Hence the capital stock declines through depreciation over \((0, T)\), as shown in panel (b). The firm continues paying a dividend for about 18 months, as shown in panel (c) shows. Indeed the dividend is higher than before date 0, since no funds are needed for investment. After about 18 months the dividend drops to zero, and the firm starts accumulating liquid assets. Panels (e) and (f) show the stocks of projects and liquid assets, and panels (d), (g), and (h) show the marginal values of these stocks.
At date $T$, after the realization of the new tax rate, there is a discrete adjustment. For the high realization, $\hat{\tau} = \tau^H$, the firm’s long-run goal involves decumulating even more capital. In the short run, it uses the entire stock of projects, $\hat{n} = \hat{m}_T$ and chooses an intensity $\hat{i}_T$ that exactly exhausts the liquid assets, so $\hat{m}_T g(\hat{i}_T) = \hat{a}_T$. Hence no discrete dividend is paid, $\hat{D} = 0$. Thereafter the firm adjusts smoothly towards the new (much lower) steady state.

For the low realization, $\hat{\tau} = \tau^H$, the firm’s long-run goal involves rebuilding its capital stock to almost the old steady state level. In the short run the discrete investment uses only part of the stock of projects $\hat{n} = \hat{m}_T$, but a higher intensity $\hat{i}_T$. Again, liquid assets are exhausted, $\hat{m}_T g(\hat{i}_T) = \hat{a}_T$, and no discrete dividend is paid, $\hat{D} = 0$. The remaining stock of projects is used over about 6 months, with investment at an intensity very similar to $\hat{i}$. Over this interval no dividend is paid. Thereafter the firm adjusts smoothly towards the new steady state. Figure 6, which shows the capital stock and expenditure on investment over a longer period, shows the prolonged adjustment to a large tax increase.

5. STOCHASTIC ARRIVAL DATE $T$

Suppose the date of the tax change is also stochastic. In particular, assume the arrival is Poisson, with arrival rate $\theta$. The firm’s problem after the arrival of the tax change is exactly as before, in (9)-(11). Before the arrival its problem is to choose $\{(D, i, n)\}_{t=0}^{\infty}$ to solve

$$\max \int_0^{\infty} e^{-(\rho+\theta)t} \left\{ D(t) + \theta E_{\hat{\tau}} [v(s; \hat{\tau})] \right\} dt, \quad \text{s.t.} \ (11),$$

where the extra term in the objective function is the post-reform continuation value, and the extra term in the discount rate represents the probability that the tax change has not yet occurred. The Hamiltonian now includes the extra term $\theta E_{\hat{\tau}} [v(s; \hat{\tau})]$.

The first order conditions in (13) are unchanged, but the law of motion for each
costate now includes a term that picks up the expected capital gain (or loss) on the asset when the tax change occurs. Let \( T \) denote the random date—a stopping time—when the tax change arrives. Then, (14) is replaced by

\[
\dot{q}_k = (\rho + \delta) q_k - q_a (1 - \tau) \pi'(k) + \theta \{q_k - E_{\hat{\tau}} [q_{kT} (s; \hat{\tau})]\},
\]

(18)

\[
\dot{q}_a \leq (\rho - r) q_a + \theta \{q_a - E_{\hat{\tau}} [q_{aT} (s; \hat{\tau})]\}, \quad \text{w/ eq. if } a > 0,
\]

\[
\dot{q}_m \leq \rho q_m + \theta \{q_m - E_{\hat{\tau}} [q_{mT} (s; \hat{\tau})]\}, \quad \text{w/ eq. if } m > 0,
\]

where \( q_{xT} (s; \hat{\tau}) \) denotes the initial value of the costate at \( T \), conditional on the new tax rate \( \hat{\tau} \), and we have used the fact that \( v_s(s; \hat{\tau}) \equiv q_{xT} (s; \hat{\tau}) \).

After \( T \) the transition is as before, although the initial condition now depends on the realization of \( T \). Over the (random) interval \( (0, T) \) the system asymptotically approaches a SS, call it \( s^* = (k^*, a^*, m^*) \). If the stopping time happens to arrive quickly, the initial condition is close to \( s(0) = (k^{**}(\tau), 0, 0) \), so the transition is essentially from one SS to another, with negligible initial stocks of assets or projects. If the stopping time happens to arrive slowly, the initial condition for the post-arrival transition is close to \( s^* \).

**a. The pre-arrival SS**

In the pre-arrival SS the firm may hold stocks of cash and projects, but it is no longer adding to those stocks. It is investing to maintain the current capital stock, and paying the rest of its profit flow as a dividend. Hence (13) implies the SS values for the costates are \( q_a^* = 1 \) and \( q_k^* = g'(i^*) \), where \( n^* = \mu \) and \( i^* = \delta k^*/\mu \). Then (18) requires

\[
\theta E_{\hat{\tau}} [q_{kT} (s^*; \hat{\tau})] = (\theta + \rho + \delta) g'(i^*) - (1 - \tau) \pi'(k^*),
\]

(19)

\[
\theta E_{\hat{\tau}} [q_{aT} (s^*; \hat{\tau})] \leq (\theta + \rho - r), \quad \text{w/ eq. if } a^* > 0,
\]

\[
\theta E_{\hat{\tau}} [q_{mT} (s^*; \hat{\tau})] \leq (\theta + \rho) q_m^*, \quad \text{w/ eq. if } m^* > 0.
\]
As a check, note that for $\theta = 0$, these conditions hold for $s^* = (k^{ss}(\tau), 0, 0)$, the benchmark SS for the initial tax rate $\tau$. If $F$ puts unit mass on $\hat{\tau} = \tau$, so the arrival entails no change, then these conditions hold at the benchmark SS for any $\theta$. If $F$ puts unit mass on $\hat{\tau} \neq \tau$, so the change is deterministic, then as $\theta \to \infty$, they hold for $s^* = (k^{ss}(\hat{\tau}), 0, 0)$, the benchmark SS for the new tax rate $\hat{\tau}$.

b. A tax change with known $\hat{\tau}$

Consider first the case where $F$ puts unit mass on $\hat{\tau} \neq \tau$, so the new tax rate is known. To see whether liquid assets or projects are accumulated, conjecture that $s^* = (k^*, 0, 0)$, and ask whether (19) holds for some $k^*$. Consider the first line in (19).

For $\theta > 0$ the LHS is positive and is strictly decreasing in $k^*$, asymptotically approaching zero as $k^* \to \infty$. The RHS (with $i^* = \delta k^*/\mu$) is strictly increasing in $k^*$. It is negative for $k^*$ sufficiently small and increases without bound as $k^* \to \infty$. Hence there is a unique solution $k^* > 0$. It is easy to verify that $k^*$ lies between $k^{ss}(\tau)$ and $k^{ss}(\hat{\tau})$. For fixed $\theta$, an increase in $\tau$ or $\hat{\tau}$ reduces $k^*$. In addition, for $\hat{\tau} < \tau$ an increase in $\theta$ increases $k^*$, and for $\hat{\tau} > \tau$ an increase in $\theta$ reduces $k^*$.

For $a^* = 0$ and $m^* = 0$, the second and third lines in (19) are inequalities. The second line holds if $k^*$ is in or sufficiently close to the region where $D > 0$ for $\hat{\tau}$, so $q_{aT} = 1$. Hence it holds if $\hat{\tau} > \tau$. It also holds if $\hat{\tau} < \tau$, provided the change is not too large. The third line also holds if $\hat{\tau} > \tau$, since in this case the marginal value of a project falls when the tax change arrives. If $\hat{\tau} < \tau$, the third line holds if the tax change is not too large, but it may fail for a large tax cut. Thus, for large tax cuts, either condition or both may fail. In these cases (presumably) inputs are held in the pre-arrival SS.

Whether inputs are accumulated also depends on $\theta$. For $\theta$ large, $k^*$ is close to $k^{ss}(\hat{\tau})$, so $q_{aT}((k^*, 0, 0); \hat{\tau}) = 1$ and $q_{mT}((k^*, 0, 0); \hat{\tau})$ is close to $q_{mT}$. Hence the second and third lines in (19) hold. For $\theta$ small, $k^*$ is close to $k^{ss}(\tau)$, which determines $q_{aT}(s^*; \hat{\tau})$.
and $q_{mT}(s^*; \hat{\tau})$. Thus, the second and third lines in (19) also hold as $\theta \to 0$. Hence it seems most likely that inputs are acquired in the case of a large tax decrease and an intermediate value for $\theta$.

c. A tax change with uncertainty about $\hat{\tau}$

Next consider the case where the new tax rate is also unknown. As an analog of Proposition 3 we have the following.

**Proposition 4:** Suppose a tax change, drawn from the distribution $F$, arrives with hazard rate $\theta > 0$. Unless $F$ puts unit mass at a single point, for all $\theta$ sufficiently large, the pre-reform SS $s^* = (k^*, a^*, m^*)$ has $k^*, a^*, m^* > 0$.

6. CONCLUSIONS

The positive predictions of the option model are very stark: policy uncertainty leads to a sharp swings in both investment and dividends. In this sense it may provide a useful framework for looking at firm-level investment in periods of high uncertainty.

To incorporate this model of firm-level investment into a macro model, it would be useful to let stored projects depreciate. There are two interpretations: that the market changes, making the investment less profitable, or that a rival firm gets access to the project and exploits it. If depreciation rates vary across projects, those with high depreciation rates are less storable. Thus, for a policy reform with a given level of uncertainty, projects with depreciation rates below a certain threshold would be stored, while those above the threshold would be exploited immediately.

In highly competitive sectors, presumably the ‘depreciation’ due to rivals is greater, making delay less feasible. Thus, highly competitive sectors should behave more like the benchmark model, and sectors with more products that are more strongly differentiated should behave more like the options model.
The welfare implications in a macroeconomic setting are less clear. In the model here, the decline in investment during the period of delay is largely offset by a boom after the uncertainty is resolved. But the same is true in many models of investment over the business cycle, so the welfare costs here might be similar to the costs of cyclical fluctuations.

[To be added: The value of the firm is a weighted sum of the values of the installed capital, liquid assets, and stored projects that it holds. The model describes the marginal values of these three stocks, and the changes in firm valuation that accompany delay might provide a useful signal about when delay is occurring.]
REFERENCES


APPENDIX A: PROOFS OF PROPOSITIONS

PROOF OF PROPOSITION 2: It suffices to show that \( \hat{m}_T > 0 \) implies \( \hat{a}_T = 0 \).

Fix \( \tilde{\tau} \) and suppose \( \hat{m}_T > 0 \). If \( n(T+z)i(T+z) = 0 \) for \( z \in (0, \Delta) \), then \( \hat{k}(T+z) < 0 \), which implies \( k > k^{ss}(\tilde{\tau}) \). In this region \( SM_0 \) lies below \( \chi(., \tilde{\tau}) \), so \( q_a = 1 \) and \( a = 0 \). Hence the solution requires \( \hat{a}_T = 0 \).

If \( n(T+z; \tilde{\tau})i(T+z; \tilde{\tau}) > 0 \) for \( z \in (0, \Delta) \), then the second and third lines in (13) hold with equality over \( (T, T+\Delta) \). Differentiate them to get two equations involving \( \dot{i} \),

\[
\frac{\dot{q}_k}{q_k} = \frac{\dot{q}_a}{q_a} + \frac{g''i}{g'}, \quad \frac{\dot{q}_m}{q_m} = \frac{\dot{q}_a}{q_a} + \frac{g''i}{g' - g/i}.
\]

Substitute from (14), using the fact that \( a > 0 \) implies \( \dot{q}_a/q_a = \rho - r \), to get

\[
(r + \delta) g' - (1 - \tilde{\tau}) \pi' = g''i, \tag{20}
\]

\[
r[g' - g/i] = g''i.
\]

If both conditions hold, then

\[
(1 - \tilde{\tau}) \pi' = \delta g' + r \frac{g}{i}.
\]

Suppose this condition holds at \( T \). It continues to hold on \( (T, T+\Delta) \) if and only if

\[
(1 - \tilde{\tau}) \pi''k = \left[ \delta g'' + \frac{r}{i} \left( g' - \frac{g}{i} \right) \right] i. \tag{21}
\]

The term in brackets on the right in (21) is positive, and the second line in (20) implies \( \dot{i} \geq 0 \). Since \( \pi'' < 0 \), (21) holds only if \( \dot{k} \leq 0 \), which implies \( k > k^{ss}(\tilde{\tau}) \). The rest of the argument is as before. ■

PROOF OF PROPOSITION 3: Suppose to the contrary that for any \( \Delta > 0 \), the optimal policy has \( n(t)i(t) > 0 \), all \( t \in [T - \Delta, T) \). Consider the following perturbation
to the conjectured solution over \((T - \Delta, T + \Delta)\), where \(\Delta > 0\) is small. Reduce the flow of projects by \(\varepsilon > 0\) over \((T - \Delta, T)\) and accumulate the projects and cash. For \(\varepsilon > 0\) sufficiently small, this is feasible. Then increase the flow of projects by \(\varepsilon\) over \((T, T + \Delta)\), and adjust the intensity on an additional group of projects of size \(\varepsilon\). For each \(\hat{\tau}\), choose the intensity for this group of \(2\varepsilon\) projects as follows.

Let \(i(T)\) denote the intensity on \((T - \Delta, T)\) and let \(i_T(\hat{\tau})\) denote the intensity on \((T, T + \Delta)\), conditional on \(\hat{\tau}\). Since \(\Delta\) is small, these intensities are approximately constant before and after \(T\), although the latter varies with \(\hat{\tau}\). For the \(2\varepsilon\) projects use the intensity
\[
i_P(\hat{\tau}) = \frac{1}{2} [i(T) + i_T(\hat{\tau})],
\]
so the capital stock at \(k(T + \Delta)\) is unaltered.

The perturbation changes the investment cost by
\[
\Delta_C(\hat{\tau}) = \varepsilon \Delta \{2g [i_P(\hat{\tau})] - g [i(T)] - g [i_T(\hat{\tau})]\}, \quad \text{all } \hat{\tau}.
\]
Since \(g\) is strictly convex, \(\Delta_C(\hat{\tau}) \leq 0\), with equality if and only if \(i_T(\hat{\tau}) = i(T)\). Unless \(F\) puts unit mass at a single point, this condition must fail on a set of \(\hat{\tau}\)'s with positive probability. Therefore, unless \(F\) puts unit mass on a single point,
\[
X \equiv \mathbb{E}_\hat{\tau}\{2g [i_P(\hat{\tau})] - g [i(T)] - g [i_T(\hat{\tau})]\} < 0.
\]
Since the perturbation reduces the cost of investment, at least weakly, for every \(\hat{\tau}\), and delays the timing of expenditures, it is also feasible in the sense that it can be financed without any additional liquid assets.

The cost of the delay is the foregone revenue. The perturbation changes the capital stock by
\[
\Delta_k(T - z) \approx -\varepsilon (\Delta - z) i(T), \quad z \in (0, \Delta),
\]
\[
\Delta_k(T + z; \hat{\tau}) \approx -\varepsilon \Delta i(T) + \varepsilon z [2i_P(\hat{\tau}) - i_T(\hat{\tau})]
= -\varepsilon (\Delta - z) i(T), \quad z \in (0, \Delta),
\]

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where the changes after $T$ are conditional on $\hat{\tau}$. Hence the change in revenue is

$$
\Delta \Pi (\hat{\tau}) \approx - (1 - \hat{\tau}) \pi' \left[ \int_0^\Delta \Delta_k (T - z) dz + \int_0^\Delta \Delta_k (T + z) dz \right]
$$

$$
\approx -2 \varepsilon (1 - \hat{\tau}) \pi' i(T) \int_0^\Delta (\Delta - z) dz
$$

$$
= - \varepsilon \Delta^2 (1 - \hat{\tau}) \pi' i(T).
$$

The reduction in revenue is of order $\varepsilon \Delta^2$, while the reduction in investment costs is of order $\varepsilon \Delta$. As noted above, $X < 0$. Hence for $\Delta > 0$ sufficiently small,

$$
E_{\hat{\tau}} [\Delta \Pi (\hat{\tau}) - \Delta_C (\hat{\tau})] \approx \varepsilon \Delta [- \Delta \pi' i(T) - X] > 0,
$$

and the perturbation raises expected profits. ■

**Proof of Proposition 4:** Suppose to the contrary that $a^* = 0$ or $m^* = 0$ or both. Let $T_0$ be a date before the arrival of the tax change, with $s(T_0) = s^*$, and let $T$ denote the (random) arrival date. Consider the following perturbation to the strategy of keeping $s(t) = s^*$ on $(T_0, T)$.

Let $i^*$ be the SS intensity, and to simplify notation, let $T_0 = 0$. Choose $\varepsilon, \Delta > 0$ small. Over $(0, \Delta)$, reduce the flow of projects by $\varepsilon$, keeping the intensity unchanged. At $t = \Delta$, the capital stock is reduced by $\varepsilon \Delta i^*$, and the firm has a stock of $m = \varepsilon \Delta$ untapped projects and a stock of $a = \varepsilon \Delta g(i^*)$ liquid assets. Over $(\Delta, T)$ reduce the intensity of replacement investment by $\varepsilon \Delta i^* \delta / \mu$, so the capital stock remains constant. Pay the interest on the accumulated liquid assets and the savings in replacement cost as dividends. The EDV of the additional dividends is

$$
\Delta_D = \varepsilon \Delta \left[ r g(i^*) + \frac{\delta}{\mu} i^* g'(i^*) \right] \int_0^\infty e^{-(\rho + \theta)t} dt
$$

$$
= \left[ r g(i^*) + \frac{\delta}{\mu} i^* g'(i^*) \right] \frac{\varepsilon \Delta}{\rho + \theta}.
$$

(22)

These terms are positive and have order $\varepsilon \Delta$. 

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After the tax change arrives, over \( (T, T + \Delta) \) increase the scale of investment by \( \varepsilon \) and alter the intensity for an additional \( \varepsilon \) projects as in the proof of Proposition 3, so the capital stock at \( T + \Delta \) is as it would have been under the original plan. Define \( i_T(\hat{\tau}) \) and \( i_P(\hat{\tau}) \) as in the proof of Prop. 3. Conditional on the new tax rate \( \hat{\tau} \), the perturbation to the investment cost is

\[
\Delta_C(\hat{\tau}) = \varepsilon \Delta \{2g [i_P(\hat{\tau})] - g(i^*) - g [i_T(\hat{\tau})]\}, \quad \text{all } \hat{\tau}.
\]

As shown in the proof of Prop. 3, \( \Delta_C(\hat{\tau}) \leq 0 \), with equality if and only if \( i_T(\hat{\tau}) = i^* \). Therefore, unless \( F \) puts unit mass on a single point,

\[
X(i^*) \equiv E_{\hat{\tau}} \{2g [i_P(\hat{\tau})] - g(i^*) - g [i_T(\hat{\tau})]\} < 0.
\]

Hence this contribution of the perturbation to the EDV of profits is

\[
-\Delta_C = -\varepsilon \Delta X(i^*) \int_0^\infty \theta e^{-(\rho+\theta)t} dt = -\theta X(i^*) \frac{\varepsilon \Delta}{\rho + \theta}.
\]

This term is positive and has order \( \varepsilon \Delta \).

The perturbation changes the capital stock by

\[
\Delta_k(t) = \begin{cases} 
-\varepsilon i^* t, & t \in (0, \Delta), \\
-\varepsilon i^* \Delta, & t \in (\Delta, T), \\
\varepsilon i^* [t - (T + \Delta)], & t \in (T, T + \Delta).
\end{cases}
\]

By assumption, the original solution can have no DA at \( T \). Hence \( k(t; \hat{\tau}) \approx k^*, \ t \in (T, T + \Delta), \ \text{all } \hat{\tau}, \) and marginal revenue \( \pi'(k^*) \) is approximately constant. The EDV of the change in revenues after \( T \), evaluated at \( T \), is

\[
\Delta_{Ra} = \text{E}_{\hat{\tau}} [(1 - \hat{\tau})] \pi' i^* \varepsilon \int_T^{T+\Delta} [t - (T + \Delta)] e^{-\rho(t-T)} dt \\
\approx -\text{E}_{\hat{\tau}} [(1 - \hat{\tau})] \pi' i^* \varepsilon \frac{\Delta^2}{2},
\]

(24)
which has order $\varepsilon \Delta^2$. The PDV of the change in revenues over $(0, \Delta)$ also has order $\Delta^2$, so both terms can be dropped.

The EDV of the change in revenue over $(\Delta, T)$ is

$$\Delta_{Rb} = -\varepsilon \Delta i^* (1 - \tau) \pi' \int_0^\infty e^{-(\rho + \theta)\theta} dt \approx -(1 - \tau) \pi' i^* \frac{\varepsilon \Delta}{\rho + \theta}. \quad (25)$$

This term is negative and has order $\varepsilon \Delta$.

Summing the components in (22)-(25) and dropping those of order higher than $\varepsilon \Delta$, we find that the perturbation is profitable for $\Delta$ sufficiently small if and only if

$$0 < rg(i^*) + \frac{\delta}{\mu} i^* g'(i^*) - \theta X(i^*) - (1 - \tau) \pi' i^*.$$

The first three terms are positive, but the last is negative. But as $\theta$ grows without bound, with $F$ fixed, $k^*$ and $i^* = k^* \delta / \mu$ converge to some limiting values. Hence for $\theta$ sufficiently large, the third term, which is positive, dominates. ■

**APPENDIX B: COMPUTATION METHOD**

This Appendix describes the computational methods used for the simulations.

**REGION A, LOW $\tau$:** The solution can be constructed by backward shooting. Choose a candidate terminal value $(k, q_k)$ on $SM_0(\hat{\tau})$ near $k^{ss}(\tau)$ and a candidate value $\Delta_T > 0$. The stock of projects is $m = 0$ at $(k, q_k)$, and the costate for projects is $q_m = q_k [i - g(i) / g'(i)]$. Construct trajectories for $(i, n)$ and $(k, m, q_k, q_m)$ by running backward for $\Delta_T$ units of time the system of ODEs

$$\begin{pmatrix} \dot{k} \\ \dot{n} \\ \dot{q}_k/q_k \\ \dot{q}_m/q_m \end{pmatrix} = - \begin{pmatrix} ni - \delta k \\ \mu - n \\ \rho + \delta - (1 - \hat{\tau}) \pi'(k)/g'(i) \\ \rho \end{pmatrix}, \quad (26)$$
with investment satisfying

\[ i - \frac{g(i)}{g'(i)} = \frac{q_m}{q_k}, \quad (27) \]

\[ n = \frac{(1 - \hat{\tau}) \pi(k)}{g(i)}. \]

The endpoint from this exercise is a candidate for \( \left( \hat{k}_T, \hat{m}_T \right) \). Let \( \hat{i} \) be the investment intensity at this point, and let \( \hat{n} = m_T - \hat{m}_T \). Check whether

\[ \hat{k}_T - k_T = \hat{n} \hat{i}, \quad \text{and} \quad a_T = \hat{n} g(i). \]

Adjust the candidate value \((k, q_k)\) on \( SM_0(\hat{\tau}) \) and time interval \( \Delta_T \) until both conditions are satisfied.

With time running forward, \( q_m \) is rising and \( q_k \) is falling, so \( i \) is rising. Hence \( q_a = q_k / g'(i) \) is falling, so \( \hat{q}_a / q_a < \rho - r \). In addition, since \( q_a \geq 1 \) on \( SM_0(\hat{\tau}) \), it follows that \( q_a \geq 1 \) on all of \((T, T + \Delta_T)\).

**REGION B, MODERATE \( \hat{\tau} \):** Let \( i^{SM}(k; \hat{\tau}) \) denote the investment intensity on \( SM_0(\hat{\tau}) \), given the capital stock \( k \). For the DA choose the intensity \( \hat{i} \) satisfying

\[ \hat{i} = i^{SM}(k_T + m_T \hat{i}; \hat{\tau}). \]

Since \( i^{SM} \) is decreasing in its first argument, there is a unique solution. And since \( i^{SM} \) is also decreasing in \( \hat{\tau} \), the solution \( \hat{i} \) is decreasing in \( \hat{\tau} \). At the threshold value for \( \hat{\tau} \) separating region B from region A, the entire stock of liquid assets is used by the initial investment, \( \hat{n} g(i) = m_T g(i) = a_T \). For higher values of \( \hat{\tau} \) the excess cash is paid out in the initial dividend \( \hat{D} \).

After the initial investment, the rest of the transition to the new SS follows \( SM_0 \), with \( n(t) \equiv \mu, m(t) = a(t) \equiv 0, q_a(t) \equiv 1, g'(i) = q_k \), and

\[ q_m = g'(i) \hat{i} - g(i). \]
The solution must also satisfy \( \dot{q}_m/q_m \leq \rho \), where
\[
\dot{\frac{q_m}{q_m}} = \frac{ig''\dot{i}}{ig'(i) - g(i)}.
\]
If \( \hat{k}_T \leq k^{ss} \), then \( \dot{q}_k \leq 0 \). Hence \( \dot{i} \leq 0 \), and the required condition holds. If \( \hat{k}_T > k^{ss} \), then \( \dot{q}_k > 0 \), and the required condition holds only if \( \hat{k}_T(\hat{\tau}) \) is not too far above \( k^{ss}(\hat{\tau}) \).

The boundary separating regions B and C is determined by the restriction on \( \dot{q}_m/q_m \).

**Region C, high \( \hat{\tau} \):** As in Region A, the transition over \( (T, T + \Delta_T) \) can be constructed using backward shooting. Choose a candidate terminal point \( (k, q_k) \) on \( SM_0(\hat{\tau}) \), with \( k \geq k^{ss}(\hat{\tau}) \), and candidates for \( \Delta_T > 0 \) and \( \{n(t), t \in (T, T + \Delta_T)\} \).

Calculate \( q_m \) as before, and construct solutions for \( i \) and \( (k, m, q_k, q_m) \) over \( (T, T + \Delta_T) \), with the intensity \( i \) satisfying
\[
q_k = g'(i),
\]
and the ODEs in (26). The endpoint from this construction is a candidate for \( \hat{k}_T, \hat{m}_T \), with
\[
\hat{m}_T = \int_T^{T+\Delta_T} [n(t) - \mu] \, dt \leq m_T.
\]
Let \( \dot{i} \) be the intensity at this point and let \( \dot{n} = m_T - \hat{m}_T \). Check whether
\[
\hat{k}_T - k_T = \dot{n}\dot{i}, \quad \text{and} \quad a_T \leq \dot{n}g(i).
\]
In addition, the first line in (27) must hold on \( (T, T + \Delta_T) \). Adjust candidate terminal point \( (k, q_k) \) on \( SM_0(\hat{\tau}) \), time interval \( \Delta_T \), and \( \{n(t)\} \) until all conditions are satisfied.
Figure 1: phase diagram for the benchmark model, $a = 0$

- $q_k = 0$
- $D = 0$
- $\chi(k)$
Figure 2a: deterministic tax change at $T > 0$ in the benchmark model

Figure 2b: stochastic tax change at $T > 0$ in the benchmark model
Figure 3: Benchmark transition: capital stock

$\tau = 0.2$

$\tau = 0.3$

$\tau = 0.4$
Figure 4: the discrete adjustment at $T$

Region A Region B Region C

$\hat{q}_{aT}$

$\hat{D}$

$\hat{n}\hat{g}(i)$

$\hat{i}$

$\hat{m}_T$

Realized tax rate $\tau$
Fig 6a: capital stock

Fig 6b: expenditure on investment